673

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LATERAL BUCKLING OF ELASTICALLY END-RESTRAINED I BEAMS

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LATERAL BUCKLING OF ELASTICALLY END-RESTRAINED I BEAMS

W. J. Austin,[1] J.M. ASCE, S. Yegian,[2] and T. P. Tung[3]

SYNOPSIS

The purpose of this paper is to present the critical lateral loads and the corresponding critical bending stresses at midspan for prismatic I beams having equal, linear, rotational restraints at both ends. In practical structures such end restraint is commonly provided by the connections. It is of importance to determine the increase of the lateral buckling load which results from this restraint. The critical loads for end-restrained bars of rectangular cross section have been discussed by G. Winter[4] and A. R. Flint.[5]

Theory

Consider an I beam, denote the horizontal principal axis as x, the vertical principal axis as y, and the longitudinal axis as z, as shown in Fig. 1. Each flange at each end is assumed to be individually restrained against rotation about the vertical axis. The rotation of each end as a whole about the x axis is elastically restrained, and no rotation at the ends is allowed about the z axis. The beam is loaded by vertical forces acting in the plane of the web. Two loadings are considered (1) uniform load and (2) concentrated load at midspan, acting at three positions, (1) on the top flange, (2) at the centroid, and (3) at the bottom flange. The maximum stresses are assumed to be less than the proportional limit of the material. It is also assumed that the external loads remain parallel to their original direction when the points of application of these loads are displaced.

The differential equations governing the lateral buckling of a symmetrical I beam supported as described above are as follows:[6]

$$EI_y u'''' = -(\beta M_x)'' \tag{1}$$

$$-D\beta'''' + C\beta'' + qs\beta = M_x u'' \tag{2}$$

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4. "Strength of Slender Beams," by George Winter, Transactions, ASCE, Vol. 109 (1944), p. 1321.
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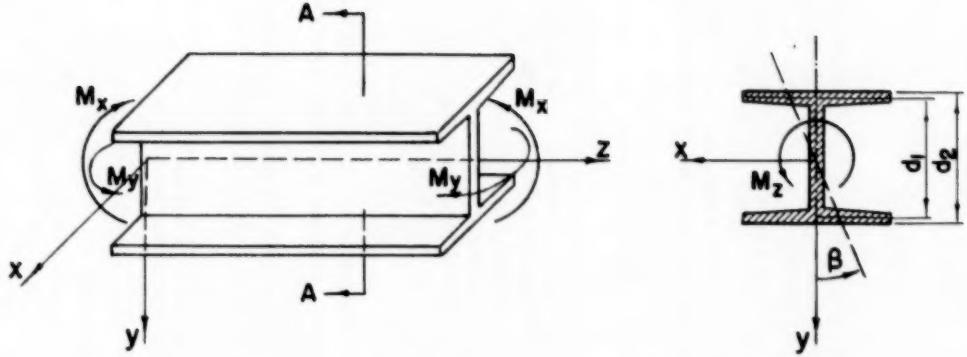
6. "Buckling Strength of Metal Structures," by F. Bleich, McGraw-Hill Book Company, Inc., New York, N. Y., 1952, p. 158.



(a) COORDINATE SYSTEM (b) SECTION A–A

FIG. 1 SIGN CONVENTION

where each prime symbol denotes one differentiation with respect to z. Other symbols used are defined as follows:

β = angle of rotation of cross-section about centroid (see Fig. 1).

u = displacement of centroid in x direction.

M_x, M_y = bending moment about x and y axes, respectively.

E = Young's modulus

I_x, I_y = moment of inertia of cross-section about x and y axes, respectively.

C = uniform torsion constant or torsional rigidity

$D = EI_y d_1^2/4$ = torsional bending constant

L = length of beam

$a = \sqrt{\frac{D}{C}}$ = section constant

q = distributed vertical load

Q = concentrated vertical load at midspan

d_1 = distance between centers of flanges, see Fig. 1

d_2 = depth of beam between outside edges of flanges, see Fig. 1

s = distance from centroid to point of load application, assumed equal to $+ d_1/2$, 0, or $- d_1/2$ for load at top flange, centroid, or bottom flange, respectively

σ = extreme fiber stress at midspan at critical load.

K_x, K_y = stiffness of end restraints against rotation about x and y axes, respectively.

N_x, N_y = dimensionless measure of end restraints, equal to $K_x L/EI_x$ and $K_y L/EI_y$, respectively.

Other symbols are defined where they appear.

Consider the origin to be at the left end of the beam. Due to symmetry we need consider only the left half of the beam. The stiffness against rotation about the x axis of the end restraint is characterized by the equation for the left end.

$$M_{xe} = -K_x (y')_{z=0} \tag{3}$$

where M_{xe} is the moment exerted on the left end of the beam by the restraint. It is convenient to use a dimensionless quantity N_x related to the stiffness K_x in the following way.

$$N_x = \frac{K_x L}{EI_x} \tag{4}$$

The flexural moments in the beam are given by the following equations:

For uniform load, $$M_x = \frac{qL^2}{2}\left[\frac{x}{L} - \frac{x^2}{L^2} - \frac{N_x}{6(2+N_x)}\right] \qquad (5a)$$

For concentrated load at midspan, $$M_x = \frac{QL}{2}\left[\frac{x}{L} - \frac{N_x}{4(2+N_x)}\right] \quad \text{with } 0 \le x \le L/2 \qquad (5b)$$

When lateral buckling occurs, the end restraints on each flange impose a lateral moment M_y on the end of each flange, as defined by the following equations,

$$M_{y1} = \frac{K_y}{2} u_1' \qquad \text{and} \qquad M_{y2} = \frac{K_y}{2} u_2' \qquad (6)$$

where the subscripts 1 and 2 denote the top and bottom flange, respectively. In Eqs. (6) the stiffness of the restraint for each flange is $K_y/2$ since the total for the beam is denoted as K_y. It is convenient to use a dimensionless quantity N_y related to the stiffness K_y in the following way:

$$N_y = K_y L/EI_y \qquad (7)$$

The curvature of each flange in the horizontal plane may be found from the ordinary beam equations, assuming that the flexural stiffness of each flange is one-half of the total beam stiffness, EI_y.

$$u_1'' = \frac{2M_{y1}}{EI_y} \qquad \text{and} \qquad u_2'' = \frac{2M_{y2}}{EI_y} \qquad (8)$$

The moments M_y may be eliminated from Eq. (8) by the use of Eq. (6)

$$u_1'' = \frac{K_y}{EI_y} u_1' \qquad \text{and} \qquad u_2'' = \frac{K_y}{EI_y} u_2' \qquad (9)$$

The lateral displacements of the top and bottom flanges may be expressed in terms of the lateral displacement of the centroidal axis, u, and the rotation, β, of the cross-section about the centroidal axis. Thus,

$$\begin{matrix} u_1 \\ u_2 \end{matrix} = u \pm \frac{d_1}{2}\beta \qquad (10a)$$

Differentiating Eq. (10a) one obtains the following:

$$\begin{matrix} u_1' \\ u_2' \end{matrix} = u' \pm \frac{d_1}{2} \beta' \tag{10b}$$

and

$$\begin{matrix} u_1'' \\ u_2'' \end{matrix} = u'' \pm \frac{d_1}{2} \beta'' \tag{10c}$$

Equations (9) may now be expressed in terms of u and β by the use of Eqs. (10b) and (10c). Thus

$$u'' \pm \frac{d_1}{2} \beta'' = \frac{K_y}{EI_y} \left(u' \pm \frac{d_1}{2} \beta' \right) \tag{11}$$

A sufficient condition for Eq. (11) to be satisfied is the following.

$$u'' = \frac{K_y}{EI_y} u' \tag{12}$$

and

$$\beta'' = \frac{K_y}{EI_y} \beta' \tag{13}$$

Equations (12) and (13) give two boundary conditions at the end of the beam. Two more are obtained from the requirement that lateral displacements and rotations at the end are prohibited. Hence,

$$u = 0 \tag{14}$$

and

$$\beta = 0 \tag{15}$$

At midspan three boundary conditions result from symmetry, as follows:

$$u' = 0, \tag{16}$$

$$u''' = 0, \tag{17}$$

and

$$\beta' = 0 \tag{18}$$

The final boundary condition at midspan depends on the loading. For distributed load,

$$\beta''' = 0 \tag{19}$$

However, for concentrated load a couple equal to $Qs\beta$ acts at midspan. Therefore, just to the left of midspan,

$$\beta''' = -\frac{Q}{2} s\beta \tag{20}$$

Equations (1) and (2) were solved numerically using a successive approximations procedure and an integration technique with twenty-four, equal divisions in the length of the beam. All numerical calculations were performed by the Illiac digital computing machine in 10 hours. The details of the numerical procedure and the coding of the digital computing machine are not given here.

Charts and Tables

In Tables 1 and 2 are given the critical values of a dimensionless load parameter for ranges of the factors L/a, N_x, and N_y. Table 1 is for a uniform load over the entire length of the beam and Table 2 is for a concentrated load at midspan. To determine the critical load, multiply the coefficients in the tables by the appropriate value of $EI_y d_1/L^4$ for uniform load, or $EI_y d_1/L^3$ for a concentrated load at midspan. The critical loads are given for three common positions of the loading, which are loaded at the top flange, centroid, and bottom flange.

The ratio L/a is the significant geometrical property of the beam for the problem of lateral buckling, somewhat like the slenderness ratio, L/r, for columns. Critical values of the load parameter are tabulated for values of the ratio L/a of 0, 1, 2, 3, 4, 5, 6, 7, 8, and 10; these values cover the range of practical interest.

Five values of each end restraint, N_x and N_y, were chosen to cover the complete range from no end restraint, N = O, to a fixed condition, N = ∞, with intermediate values roughly equally spaced in effectiveness. For example, the values of N_x were taken to give equal increments in the ratio of end moment to midspan moment, as shown in the following table.

Uniform Load		Concentrated Load at Midspan	
Restraint N_x	End Moment / Midspan Moment	Restraint N_x	End Moment / Midspan Moment
0	0	0	0
2	-0.5	4/3	-0.25
6	-1.0	4	-0.5
18	-1.5	12	-0.75
∞	-2.0	∞	-1.0

In using the tables and curves, it is often more convenient to use the concept of the ratio of end moment to center moment rather than the stiffness N_x. The values of N_y were chosen in a similar manner.

Critical stress is generally the quantity which is most useful in design. Figures 2 to 12 show the variation of a dimensionless critical stress parameter with L/a, for all combinations of the values of N_x and N_y previously discussed and the three positions of loading. The critical stress parameter gives the extreme fiber stress at midspan corresponding to the critical load. This stress is always the maximum stress for the case of a concentrated load at midspan, but, as shown in the previous table is not the maximum stress in the beam for the uniform load when $N_x > 6$.

Figures 2 to 6 are for distributed load and Figs. 7 to 12 are for concentrated load at midspan. Each figure gives all the curves which correspond to a single value of the restraint N_y, except Figs. 11 and 12 which together are complete for $N_y = \infty$.

Each figure presents the curves for five different values of the end restraint N_x and three loading positions, a total of 15 curves except where the curves lie so close together that they cannot be drawn separately. When possible, the individual curves are identified to the right of the figure where the value of N_x is written beside the terminus of the curve to which it applies. The loading position is also indicated to the right of the figure. When two curves are coincident the pertinent values of N_x are written side by side.

Example

As an example, consider a 12" I beam attached to fixed supports by riveted clip angle connections and supporting a uniform load, as shown in Fig. 13(a). The properties of the beam are as follows:

$E = 30 \times 10^6$ psi

$I_y = 9.5$ in.^{4}

$S_x = 36.0$ in.^{3}

$d_1 = 11.5$ in.

$L = 264$ in.

In booklet S-57 of the Bethlehem Steel Company, the value of the torsion constant $\underline{a}$ is given as 29.6 in. Hence,

$$L/a = 264/29.6 = 8.9$$

The moment-rotation relation for this connection as given by Rathbun[7] has been reproduced in Fig. 13(b). The curve is non-linear but this non-linearity is no essential handicap to the analysis. The restraint about the horizontal axis governs only the magnitude of the vertical end moment. This restraint does not participate in the buckling action since during buckling the vertical moments M_x are assumed to be independent of the lateral displacement and rotation.

Since the moment-rotation relation for the end connections is non-linear, the moments in the beam are not proportional to the magnitude of the uniform

7. "Elastic Properties of Riveted Connections," by J. C. Rathbun, Transactions, ASCE, Vol. 101 (1936), p. 524.

load. The flexural moments which result from any given load may be found as explained by Rathbun. By making this computation for several assumed values of the load one may obtain the curve of load vs. the moment ratio M_e/M_c which is shown in dashed line in Fig. 13(c). Note that this line is inclined downward to the right and is slightly curved. If the riveted connection were linearly elastic, this curve would be a straight vertical line. Of course, if the end connections were linearly elastic, Fig. 13(c) would be unnecessary as the critical stress could be read directly from Fig. 2. The dashed curve in Fig. 13(c) defines the flexural action of the beam on the assumption that the beam is in neutral or stable equilibrium.

The clip angle connection affords no restraint against rotation of the flanges about a vertical axis. Therefore,

$$N_y = 0.$$

It is also assumed that the clip angles prevent rotation of the beam about the centroidal axis at both ends. The curve of buckling load vs. the moment ratio M_e/M_c may now be established for this beam by interpolating between the pertinent values in Table 1. The buckling curve is shown in solid line in Fig. 13(c). Note that the moment ratio M_e/M_c is used instead of the parameter N_x; the relation between the two is defined in the previous table.

The intersection of the buckling load curve and the basic bending relation curve defines the magnitude of the critical uniform load and also the moment ratio. In Fig. 13(c), this intersection occurs at the following values:

$$\text{critical load} = w = 1180 \text{ lb. per foot}$$

$$M_e/M_c = -0.29$$

Since

$$M_c - M_e = \frac{1}{8} wL^2 = 857 \text{ in. kips},$$

then

$$M_e = -193 \text{ in kips},$$

and

$$M_c = 664 \text{ in kips}.$$

The maximum critical stress is hence

$$\sigma cr = \frac{M_c}{S_x} = \frac{664000}{36} = 18400 \text{ psi}.$$

If the beam were assumed to be simply supported at the ends, the critical load and critical stress would be as follows:

$$\text{critical load} = 930 \text{ lb./ft.}$$
$$\text{critical stress} = 18800 \text{ psi}.$$

Thus, the restraint afforded by the clip angles raises the critical load by 28 per cent for this particular case. On the other hand, the critical stress is slightly decreased.

In general, the critical maximum stress varies much less than the critical load as the restraint about the horizontal axis, N_x, changes. The critical load always increases as N_x increases, but the critical stress may either increase or decrease. The latter trend is illustrated in the curves for critical stress at midspan due to concentrated load acting on the top flange, which are shown in Figs. 7 to 11.

In applying these solutions to practical problems there is seldom any difficulty in finding the proper value of N_x, or its equivalent M_e/M_c. However, often the value of N_y can only be found by making simplifying assumptions.

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TABLE 1

CRITICAL VALUES OF $qL^4/EI_y d_1$ FOR UNIFORM LOAD

$N_x = K_x L/EI_x \qquad N_y = K_y L/EI_y$

N_y	N_x	L/a									
		0	1	2	3	4	5	6	7	8	10
		(A) Load at Top Flange									
0	0	28.7	30.7	36.3	44.6	54.7	66.0	78.0	90.5	103.4	129.9
	2	36.1	38.9	47.0	59.1	74.2	91.2	109.5	128.8	148.7	189.8
	6	40.5	44.0	53.9	69.2	88.5	110.7	134.9	160.7	187.5	243.4
	18	43.1	47.0	58.2	75.8	98.5	125.0	154.4	186.0	219.2	288.9
	∞	44.6	48.8	60.9	80.1	105.2	135.1	168.6	205.1	243.7	325.8
2/3	0	35.5	37.5	43.1	51.5	62.0	73.8	86.5	99.9	113.8	142.3
	2	44.5	47.3	55.3	67.5	82.8	100.4	119.6	139.8	160.9	204.6
	6	50.0	53.4	63.2	78.4	97.8	120.4	145.3	172.0	200.0	258.3
	18	53.3	57.1	68.1	85.5	108.1	134.8	164.6	196.8	230.8	302.7
	∞	55.2	59.3	71.2	90.1	115.0	144.8	178.4	215.1	254.2	337.7
2	0	46.4	48.3	53.9	62.5	73.2	85.6	99.2	113.6	128.5	159.7
	2	58.1	60.8	68.7	80.8	96.4	114.4	134.4	155.7	178.0	224.8
	6	65.3	68.6	78.1	93.1	112.5	135.4	160.9	188.4	217.4	278.7
	18	69.7	73.4	84.1	101.2	123.5	150.2	180.2	212.9	247.6	321.6
	∞	72.5	76.4	88.0	106.5	131.0	160.4	194.0	230.8	270.2	354.8
6	0	66.6	68.5	74.0	82.5	93.5	106.5	120.9	136.4	152.7	187.1
	2	83.5	86.1	93.7	105.6	121.2	139.6	160.3	182.6	206.3	256.5
	6	94.2	97.3	106.5	121.0	140.2	163.1	189.0	217.2	247.2	311.3
	18	101.0	104.5	114.8	131.4	153.3	179.6	209.6	242.6	277.9	353.7
	∞	105.5	109.3	120.4	138.4	162.3	191.3	224.5	261.2	300.7	386.1
∞	0	113.8	115.6	121.0	129.6	141.0	154.7	170.3	187.3	205.5	244.5
	2	146.0	148.6	156.1	168.2	184.2	203.5	225.5	249.6	275.5	331.1
	6	168.3	171.4	180.7	195.5	215.3	239.2	266.6	296.8	329.2	399.0
	18	183.9	187.5	198.1	215.1	237.9	265.6	297.4	332.5	370.3	452.1
	∞	195.0	199.0	210.6	229.4	254.6	285.3	320.6	359.8	402.0	493.8
		(B) Load at Centroid									
0	0	44.7	46.9	53.0	61.8	72.3	84.0	96.3	109.0	122.1	148.8
	2	70.7	74.2	83.8	97.7	114.3	132.7	152.1	172.2	192.8	234.9
	6	98.4	103.3	116.6	136.0	159.1	184.6	211.6	239.5	268.1	326.5
	18	126.9	133.1	150.3	175.3	205.1	237.9	272.6	308.5	345.3	420.5
	∞	154.5	162.2	183.2	213.6	249.9	290.0	332.3	376.2	421.1	512.9
2/3	0	54.7	56.9	63.0	72.0	82.9	95.1	108.2	121.9	135.9	164.8
	2	85.4	88.8	98.2	112.2	129.3	148.3	168.7	189.9	211.8	256.7
	6	117.1	121.7	134.7	153.9	177.2	203.3	231.2	260.3	290.2	351.8
	18	148.6	154.5	171.0	195.3	225.0	258.1	293.5	330.5	368.5	446.5
	∞	178.6	185.7	205.5	234.7	270.4	310.3	352.8	397.3	443.0	536.9

(Continued on Next Page)

TABLE 1 (Continued)

N_y	N_x	L/a 0	1	2	3	4	5	6	7	8	10
2	0	70.4	72.5	78.5	87.6	98.9	111.8	125.8	140.5	155.7	187.3
	2	107.8	111.0	120.2	134.1	151.4	171.1	192.4	214.9	238.2	286.5
	6	145.0	149.4	161.7	180.4	203.7	230.2	258.9	289.1	320.4	385.3
	18	180.9	186.3	201.7	225.0	254.0	287.0	322.8	360.5	399.6	480.5
	∞	214.0	220.4	238.6	266.2	300.6	339.7	382.0	426.6	472.8	568.4
6	0	97.8	99.9	105.7	114.7	126.3	139.8	154.6	170.6	187.2	222.2
	2	146.0	149.0	157.7	171.2	188.4	208.5	230.7	254.5	279.3	331.5
	6	192.1	196.0	207.5	225.2	247.9	274.3	303.4	334.6	367.2	435.6
	18	235.0	239.9	253.8	275.5	303.2	335.4	370.9	408.9	448.7	532.0
	∞	274.1	279.7	295.9	321.1	353.3	390.7	431.9	475.9	522.0	618.4
∞	0	153.3	155.2	160.8	169.9	181.7	195.9	212.0	229.5	248.1	287.9
	2	225.1	227.9	236.2	249.4	266.7	287.5	310.9	336.4	363.6	421.5
	6	292.8	296.4	307.2	324.2	346.7	373.4	403.7	436.6	471.5	545.9
	18	355.9	360.3	373.3	393.9	420.9	453.2	489.5	529.0	571.0	660.0
	∞	414.2	419.3	434.2	458.0	489.2	526.3	568.1	613.4	661.5	763.2
		(C) Load at Bottom Flange									
0	0	69.6	71.6	77.2	85.5	95.5	106.7	118.7	131.2	144.1	170.5
	2	138.0	140.8	148.8	160.8	175.6	192.4	210.6	229.6	249.4	290.1
	6	236.5	239.9	249.7	264.6	283.5	305.2	329.0	354.3	380.7	435.8
	18	366.0	369.8	380.7	397.8	419.9	445.8	474.5	505.4	538.0	606.6
	∞	522.2	526.3	538.1	556.9	581.4	610.6	643.5	679.3	717.3	798.1
2/3	0	84.3	86.3	91.9	100.3	110.7	122.5	135.2	148.5	162.3	190.7
	2	162.8	165.6	173.5	185.5	200.7	218.1	237.0	257.0	277.9	321.3
	6	270.9	274.2	283.8	298.7	317.8	340.0	364.6	390.8	418.4	476.1
	18	407.0	410.8	421.6	438.7	460.9	487.1	516.4	548.2	581.7	652.8
	∞	565.6	569.6	581.4	600.0	624.6	654.0	687.2	723.5	762.1	844.6
2	0	106.6	108.5	114.1	122.6	133.3	145.6	159.1	173.4	188.3	219.4
	2	198.2	200.9	208.7	220.7	236.1	254.0	273.8	295.0	317.1	363.6
	6	317.5	320.7	330.2	345.0	364.2	386.9	412.2	439.4	468.2	529.1
	18	460.8	464.5	475.1	492.1	514.3	540.8	570.7	603.1	637.7	711.2
	∞	621.0	625.0	636.5	654.9	679.3	708.7	742.0	778.5	817.7	901.6
6	0	142.8	144.7	150.1	158.6	169.6	182.6	196.9	212.4	228.7	263.1
	2	252.3	254.9	262.5	274.4	290.0	308.5	329.2	351.6	375.3	425.6
	6	385.6	388.8	398.0	412.7	432.0	455.1	481.1	509.5	539.7	604.1
	18	537.8	541.4	551.8	568.5	590.7	617.3	647.6	680.7	716.2	792.2
	∞	701.1	704.9	716.1	734.1	758.0	787.0	820.2	856.7	896.0	980.7
∞	0	204.7	206.6	212.1	220.8	232.4	246.2	262.0	279.3	297.7	337.2
	2	342.4	345.0	352.8	365.2	381.6	401.4	423.9	448.7	475.2	532.0
	6	501.4	504.6	514.2	529.6	550.1	574.8	603.1	634.1	667.5	739.2
	18	677.7	681.4	692.4	710.0	733.5	762.1	794.8	830.9	869.6	953.1
	∞	865.5	869.5	881.4	900.5	926.1	957.2	992.9	1032	1075	1167

TABLE 2

CRITICAL VALUES OF $QL^3/EI_y d_1$ FOR CONCENTRATED LOAD

$N_x = K_x L/EI_x$ $\quad N_y = K_y L/EI_y$

N_y	N_x	L/a									
		0	1	2	3	4	5	6	7	8	10
		(A) Load at Top Flange									
0	0	15.8	16.9	20.2	25.0	30.8	37.4	44.4	51.8	59.4	75.0
	4/3	18.0	19.5	23.5	29.6	37.1	45.5	54.7	64.3	74.2	94.7
	4	19.7	21.3	26.0	33.2	42.2	52.5	63.7	75.6	87.9	113.5
	12	20.8	22.6	27.9	36.0	46.3	58.2	71.4	85.3	100.0	130.6
	∞	21.6	23.5	29.2	38.0	49.3	62.6	77.4	93.3	110.0	145.3
2/3	0	19.2	20.4	23.6	28.4	34.4	41.2	48.5	56.3	64.3	80.9
	4/3	21.9	23.3	27.3	33.3	40.8	49.5	58.9	68.9	79.2	100.8
	4	23.9	25.5	30.0	37.1	46.1	56.4	67.8	80.0	92.7	119.2
	12	25.2	27.0	32.1	40.0	50.1	62.0	75.2	89.3	104.2	135.6
	∞	26.2	28.0	33.5	42.0	53.1	66.3	81.0	96.8	113.7	149.4
2	0	24.6	25.7	28.8	33.6	39.7	46.7	54.4	62.5	71.0	88.8
	4/3	28.0	29.3	33.1	39.0	46.5	55.2	64.9	75.3	86.1	108.8
	4	30.4	31.9	36.3	43.1	51.9	62.3	73.8	86.1	99.2	126.7
	12	32.1	33.8	38.6	46.2	56.1	67.8	80.9	95.1	110.2	142.2
	∞	33.3	35.1	40.2	48.4	59.2	72.0	86.5	102.3	119.1	155.0
6	0	34.2	35.2	38.1	42.8	48.8	56.0	63.9	72.5	81.5	100.7
	4/3	38.6	39.8	43.4	49.0	56.4	65.1	74.9	85.6	96.8	120.8
	4	41.9	43.3	47.4	53.8	62.3	72.5	84.0	96.5	109.8	138.2
	12	44.3	45.8	50.3	57.5	66.9	78.3	91.2	105.3	120.4	152.9
	∞	46.0	47.7	52.5	60.1	70.4	82.7	96.8	112.4	129.1	165.0
∞	0	55.0	55.9	58.8	63.3	69.4	76.6	84.9	94.0	103.8	124.7
	4/3	62.4	63.6	67.0	72.5	79.8	88.6	98.7	109.8	121.7	147.4
	4	68.2	69.5	73.4	79.7	88.1	98.2	109.9	122.8	136.7	166.7
	12	72.6	74.0	78.3	85.3	94.6	106.0	119.0	133.5	149.2	183.0
	∞	76.0	77.5	82.2	89.7	99.8	112.2	126.4	142.3	159.4	196.7
		(B) Load at Centroid									
0	0	26.9	28.2	31.8	37.1	43.4	50.4	57.8	65.4	73.2	89.1
	4/3	36.2	37.9	42.8	49.9	58.4	67.7	77.6	87.8	98.3	119.6
	4	46.4	48.7	54.9	64.0	74.8	86.8	99.4	112.4	125.7	152.9
	12	57.1	59.9	67.7	78.8	92.2	106.8	122.3	138.3	154.7	188.2
	∞	67.7	71.1	80.2	93.5	109.3	126.8	145.2	164.2	183.7	223.4
2/3	0	32.5	33.8	37.4	42.7	49.2	56.4	64.2	72.2	80.5	97.6
	4/3	43.2	44.9	49.6	56.7	65.3	74.8	85.1	95.7	106.7	129.2
	4	54.5	56.7	62.7	71.6	82.4	94.5	107.4	120.8	134.6	163.0
	12	66.1	68.7	76.0	86.8	100.0	114.6	130.2	146.5	163.3	197.6
	∞	77.3	80.4	88.9	101.6	116.9	134.1	152.4	171.5	191.1	231.3

(Continued on Next Page)

TABLE 2 (Continued)

N_y	N_x	L/a 0	1	2	3	4	5	6	7	8	10
2	0	41.1	42.3	45.8	51.1	57.7	65.2	73.3	81.8	90.7	109.0
	4/3	53.6	55.2	59.7	66.6	75.2	84.9	95.5	106.6	118.1	142.0
	4	66.5	68.5	74.1	82.6	93.3	105.3	118.4	132.2	146.5	176.0
	12	79.2	81.6	88.3	98.5	111.2	125.6	141.2	157.6	174.6	209.7
	∞	91.3	94.0	101.7	113.5	128.1	144.7	162.7	181.6	201.2	241.7
6	0	55.5	56.7	60.0	65.1	71.7	79.3	87.7	96.7	106.2	125.9
	4/3	70.9	72.3	76.5	83.0	91.4	101.1	111.9	123.3	135.4	160.6
	4	86.1	87.8	92.9	100.9	111.0	122.8	135.8	149.8	164.3	194.9
	12	100.7	102.8	108.8	118.0	129.9	143.7	158.9	175.2	192.2	227.8
	∞	114.4	116.8	123.5	134.0	147.5	163.1	180.4	198.8	218.1	258.4
∞	0	84.0	85.0	88.1	93.1	99.6	107.3	116.1	125.6	135.8	157.5
	4/3	105.4	106.7	110.6	116.8	124.9	134.6	145.6	157.5	170.2	197.2
	4	126.4	128.0	132.6	140.0	149.7	161.2	174.3	188.5	203.6	235.7
	12	146.6	148.4	153.7	162.3	173.4	186.8	201.8	218.1	235.5	272.4
	∞	165.7	167.7	173.8	183.3	195.9	210.8	227.7	246.0	265.4	306.6
		(C) Load at Bottom Flange									
0	0	45.5	46.7	50.0	54.9	60.8	67.5	74.7	82.2	89.8	105.6
	4/3	71.9	73.3	77.5	83.7	91.4	100.1	109.4	119.3	129.4	150.3
	4	108.2	109.9	114.7	122.2	131.5	142.1	153.7	165.9	178.6	204.8
	12	154.8	156.7	162.2	170.7	181.5	194.0	207.7	222.2	237.4	269.0
	∞	208.3	210.4	216.5	226.0	238.2	252.4	268.1	285.0	302.6	339.6
2/3	0	54.6	55.8	59.0	63.9	70.0	76.9	84.4	92.3	100.4	117.2
	4/3	84.3	85.7	89.7	95.9	103.6	112.5	122.1	132.3	142.9	164.9
	4	123.2	124.9	129.6	136.9	146.2	156.9	168.6	181.2	194.2	221.4
	12	170.9	172.7	178.0	186.3	196.9	209.4	223.1	237.9	253.3	285.7
	∞	223.0	225.0	230.8	240.0	251.9	266.0	281.6	298.5	316.2	353.6
2	0	68.1	69.2	72.4	77.2	83.4	90.5	98.3	106.6	115.2	133.2
	4/3	101.9	103.3	107.1	113.1	120.8	129.8	139.7	150.2	161.2	184.3
	4	143.9	145.5	150.0	157.0	166.1	176.7	188.6	201.3	214.7	242.8
	12	192.5	194.2	199.2	207.2	217.5	229.7	243.3	258.1	273.6	306.6
	∞	243.0	244.9	250.4	259.1	270.5	284.1	299.4	316.0	333.6	371.0
6	0	89.9	90.9	93.9	98.6	104.7	111.9	120.0	128.6	137.8	157.1
	4/3	129.2	130.5	134.1	139.8	147.3	156.2	166.1	176.9	188.3	212.5
	4	175.2	176.6	180.7	187.3	196.0	206.4	218.0	230.8	244.3	273.1
	12	225.3	226.9	231.5	238.8	248.5	260.1	273.3	287.7	303.2	336.2
	∞	275.5	277.2	282.1	290.1	300.6	313.2	327.7	343.6	360.6	397.3
∞	0	127.9	128.9	131.7	136.3	142.4	149.8	158.2	167.4	177.2	198.4
	4/3	177.3	178.5	181.9	187.4	194.8	203.7	213.9	225.1	237.1	263.1
	4	232.9	234.2	238.1	244.4	252.8	263.0	274.8	287.7	301.6	331.7
	12	292.6	294.0	298.3	305.2	314.4	325.6	338.5	352.9	368.3	401.9
	∞	353.2	354.7	359.2	366.5	376.3	388.2	402.0	417.4	434.0	470.3

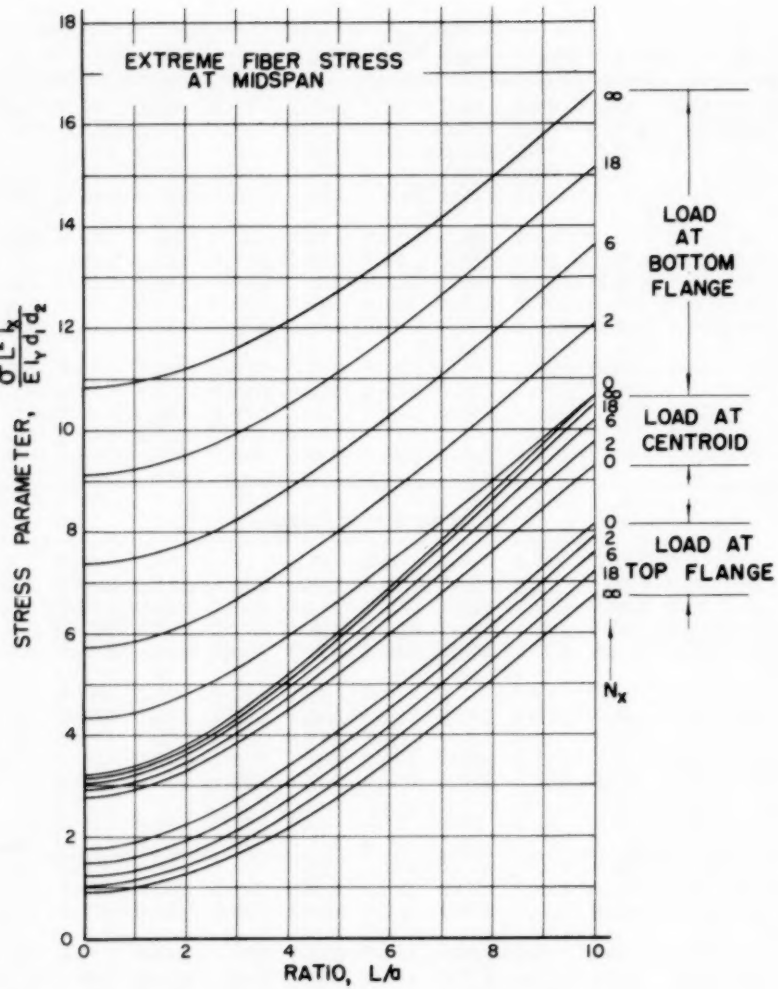


FIG. 2 CRITICAL STRESS CURVES FOR UNIFORM LOAD FOR $N_y = 0$

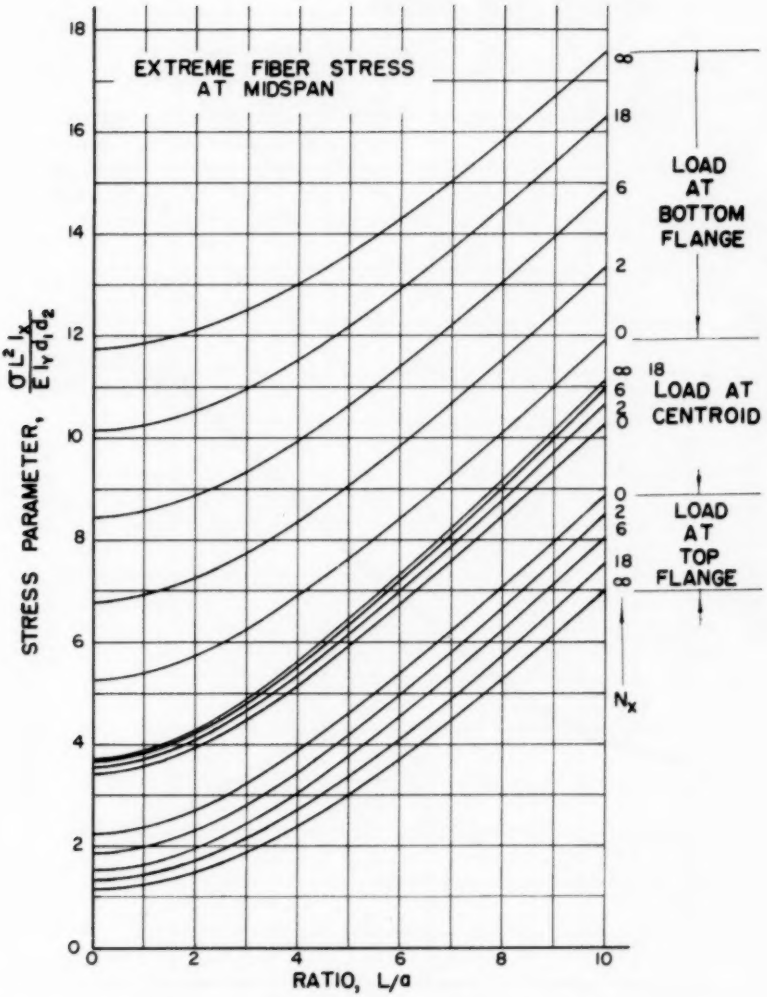


FIG. 3 CRITICAL STRESS CURVES FOR UNIFORM LOAD FOR $N_Y = 2/3$

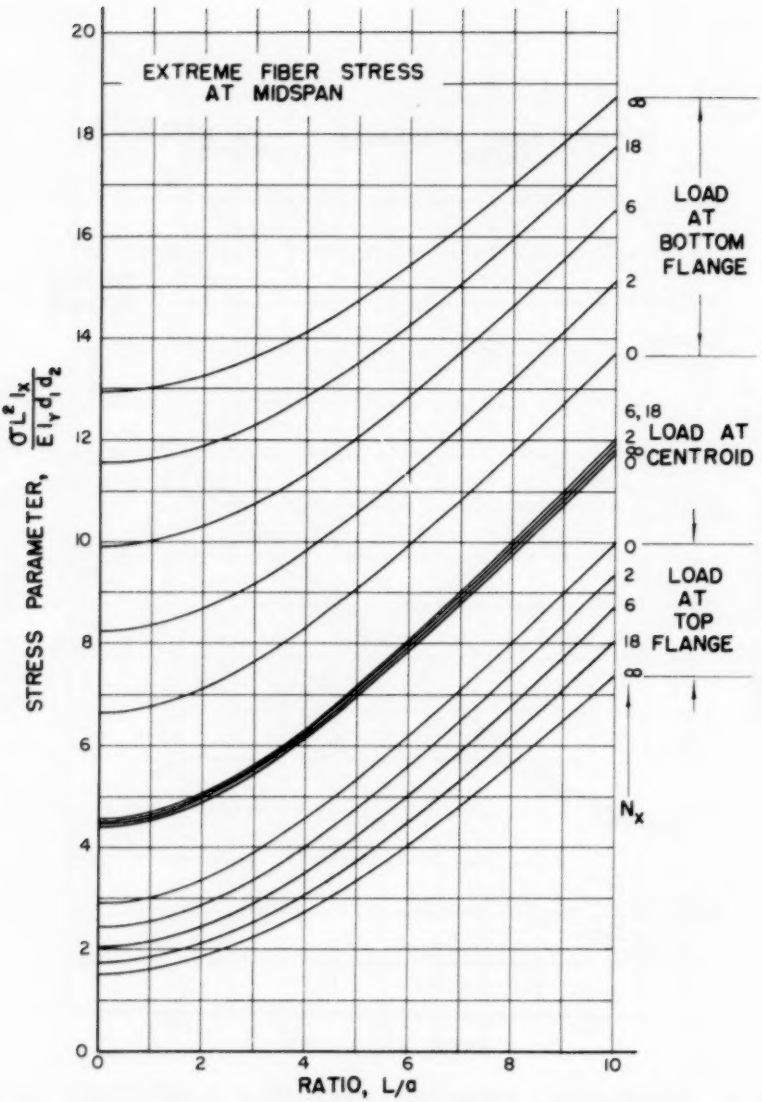


FIG. 4 CRITICAL STRESS CURVES FOR UNIFORM LOAD FOR $N_Y = 2$

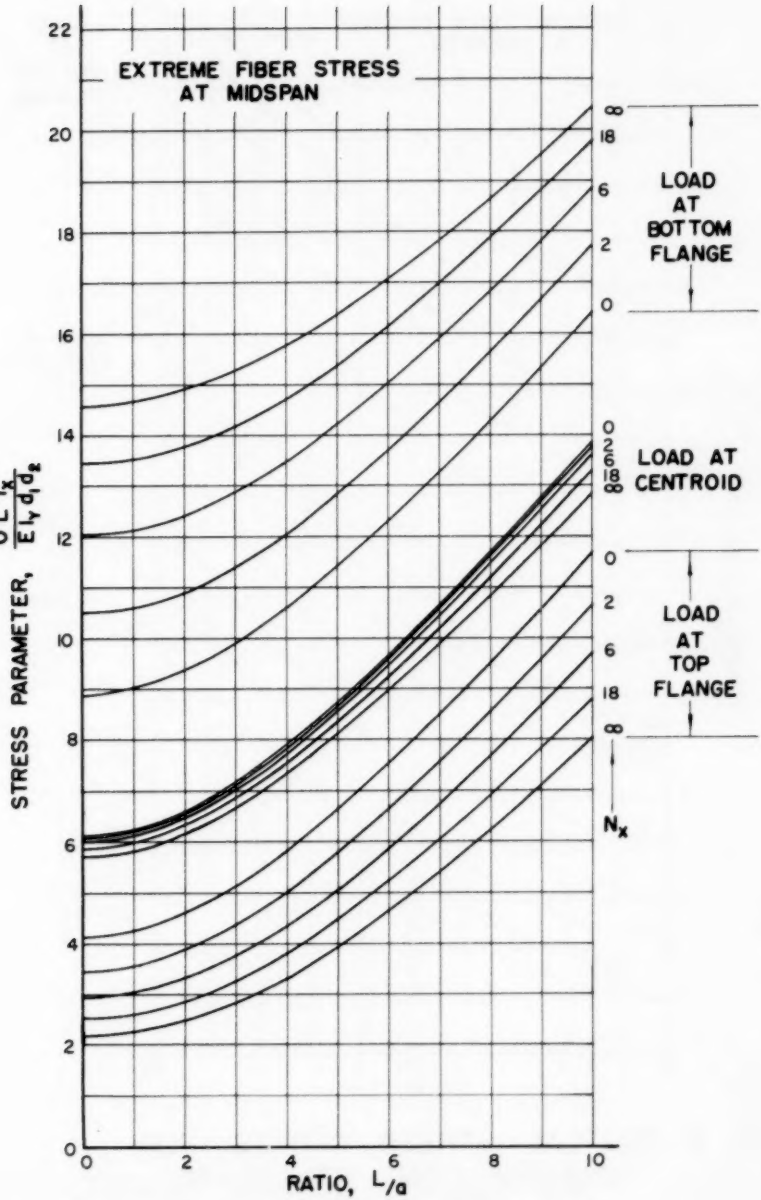


FIG. 5 CRITICAL STRESS CURVES FOR UNIFORM LOAD FOR $N_y = 6$

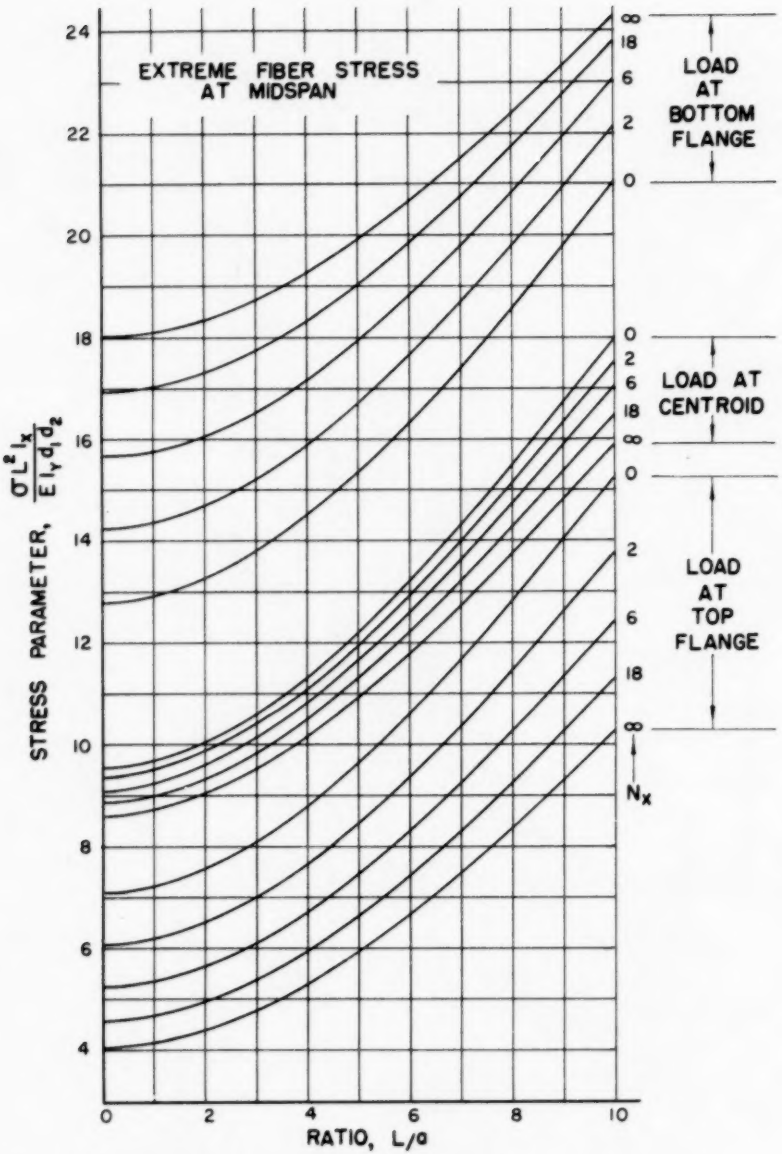


FIG. 6 CRITICAL STRESS CURVES FOR UNIFORM LOAD FOR $N_y = \infty$

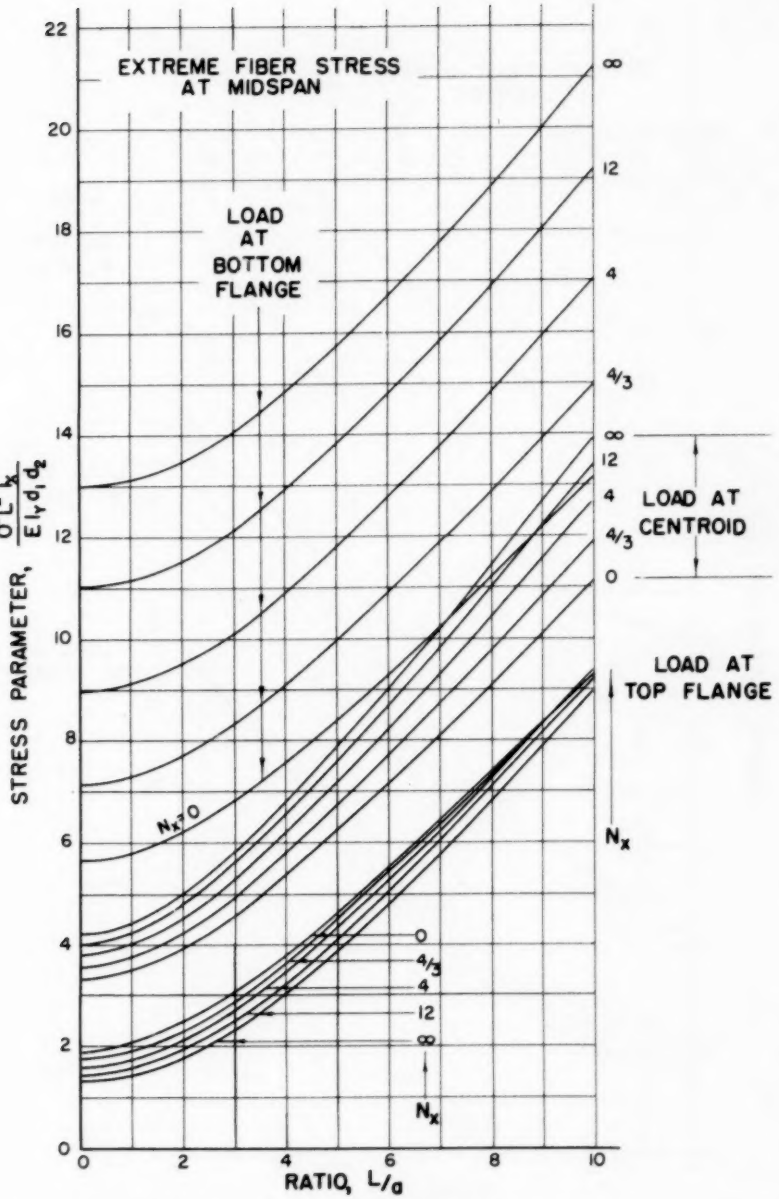


FIG. 7 CRITICAL STRESS CURVES FOR CONCENTRATED LOAD FOR $N_y = 0$

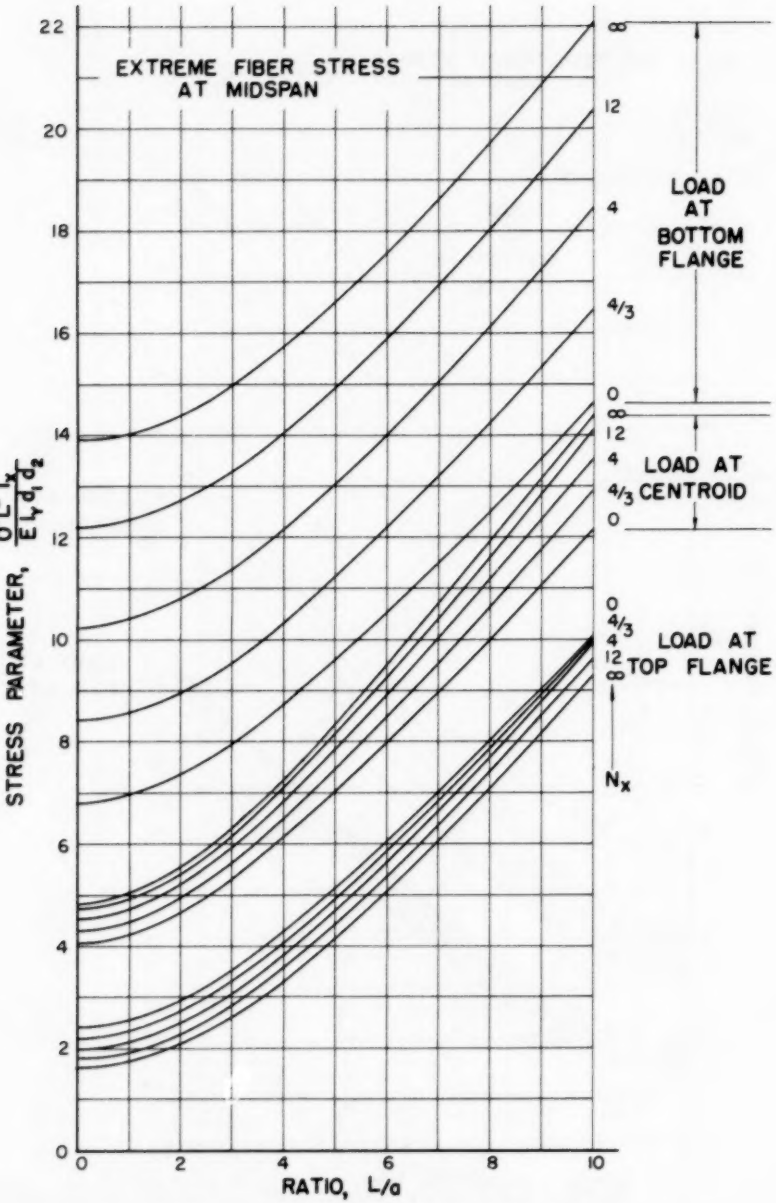


FIG. 8 CRITICAL STRESS CURVES FOR CONCENTRATED LOAD FOR $N_y = 2/3$

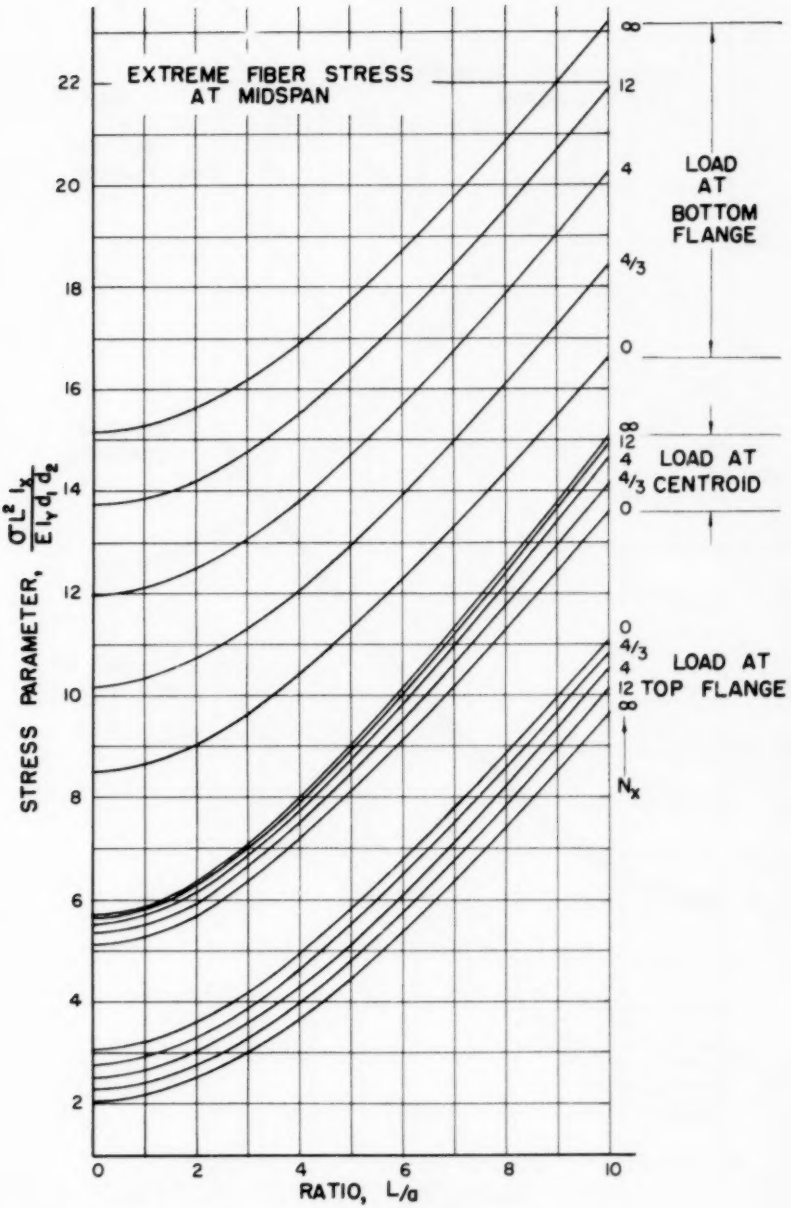


FIG. 9 CRITICAL STRESS CURVES FOR CONCENTRATED LOAD FOR $N_y = 2$

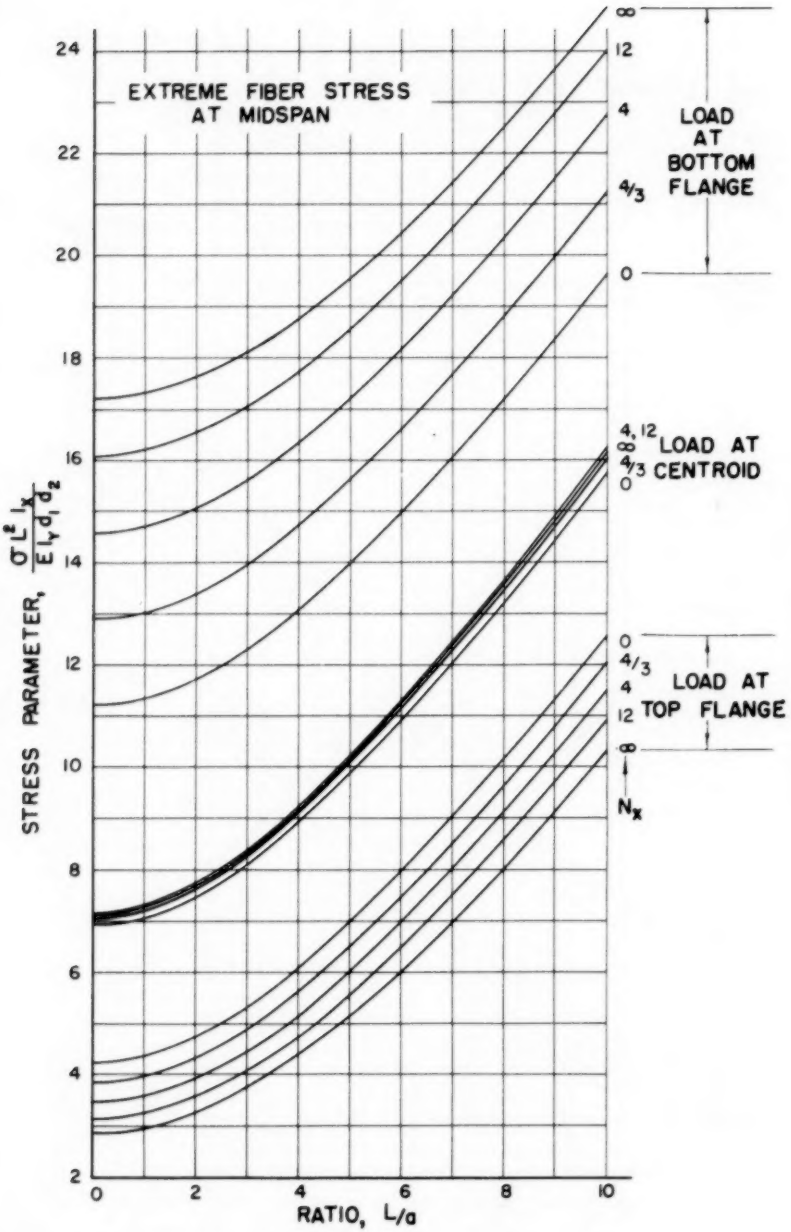


FIG. 10 CRITICAL STRESS CURVES FOR CONCENTRATED LOAD FOR $N_y = 6$

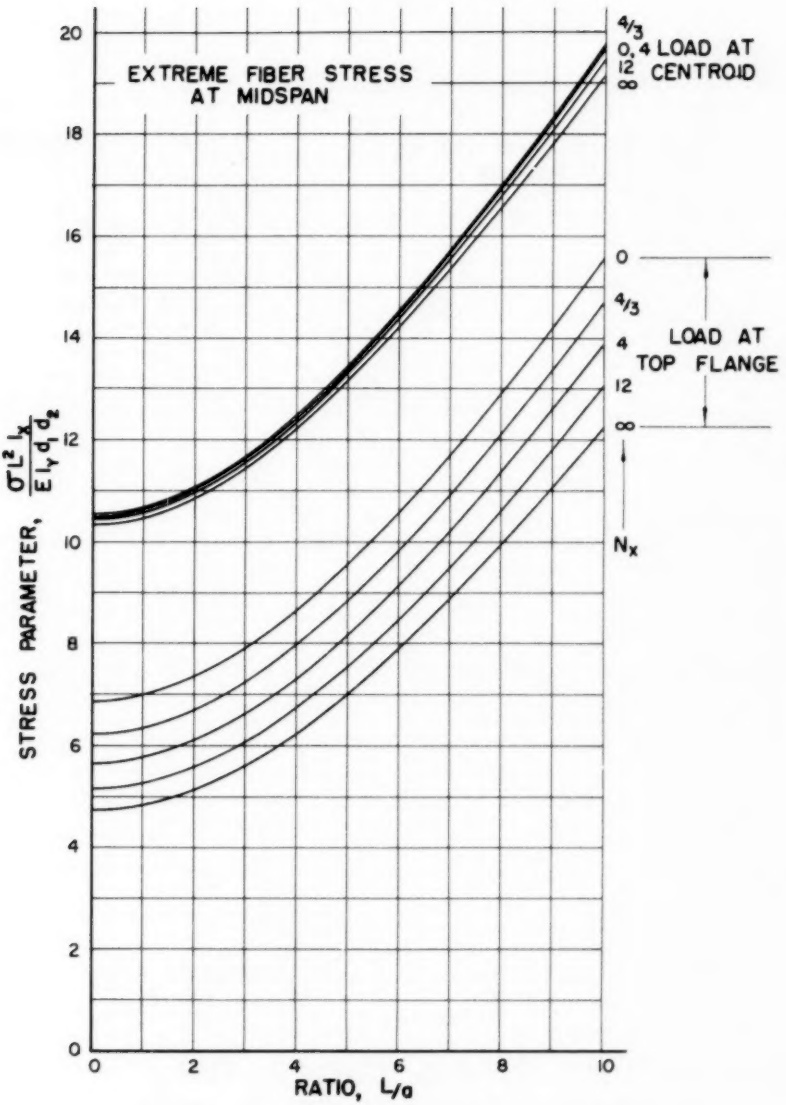


FIG. 11 CRITICAL STRESS CURVES FOR CONCENTRATED LOAD FOR $N_Y = \infty$

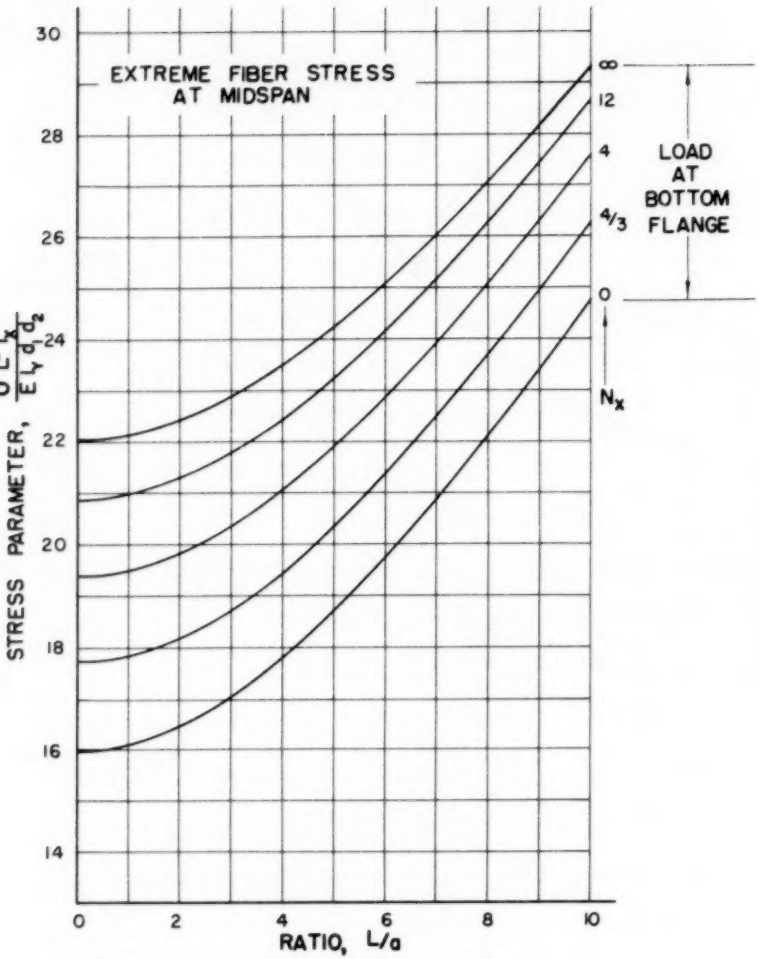
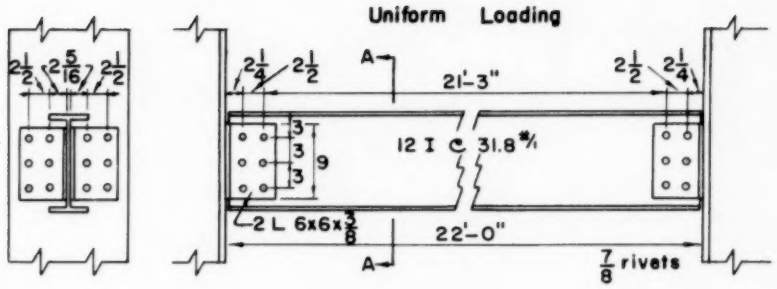
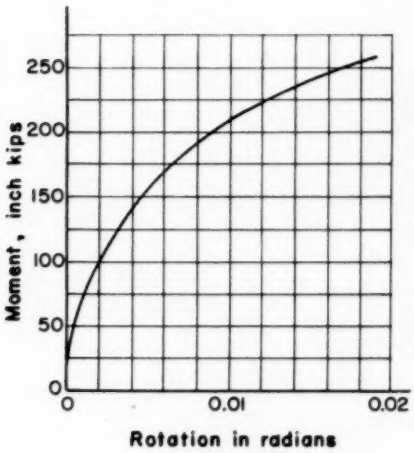


FIG. 12 CRITICAL STRESS CURVES FOR CONCENTRATED LOAD FOR $N_y = \infty$

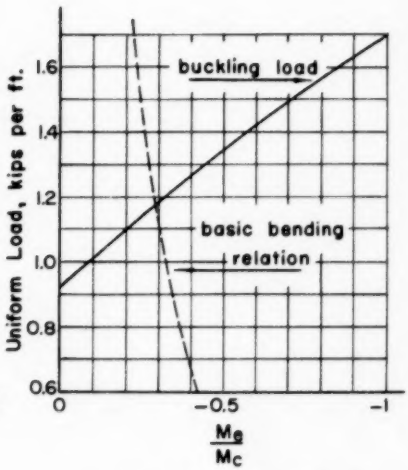


SECT. A-A

(a) BEAM WITH RIVETED CONNECTIONS



(b) MOMENT ROTATION CURVE



(c) LOAD vs RATIO OF END TO CENTER MOMENTS

FIG. 13

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